REPORT DOCUMENTATION PAGE

Form Approved OMB NO. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
31-01-2018	Final Report	7-Jul-2014 - 31-Jul-2017

4. TITLE AND SUBTITLE	
Final Report: Studies in the Control of Stochastic Systems	5a. CONTRACT NUMBER W911NF-14-1-0390
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER 611102
6. AUTHORS	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAMES AND ADDRESSES	8. PERFORMING ORGANIZATION REPORT NUMBER
University of Kansas 2385 Irving Hill Road Lawrence, KS 66045 -7568	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS (ES) U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211	10. SPONSOR/MONITOR'S ACRONYM(S) ARO 11. SPONSOR/MONITOR'S REPORT NUMBER(S) 63994-MA.16

12. DISTRIBUTION AVAILIBILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

The views, opinions and/or findings contained in this report are those of the author(s) and should not contrued as an official Department of the Army position, policy or decision, unless so designated by other documentation.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Tyrone Duncan
UU	UU	UU	UU		19b. TELEPHONE NUMBER 785-864-3032

Standard Form 298 (Rev 8/98)
Prescribed by ANSI Std. Z39.18

RPPR Final Report
as of 01-Feb-2018

Agency Code:

Proposal Number: 63994MA **Agreement Number: W911NF-14-1-0390**

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DUNS Number: 076248616 EIN: 480680117
Report Date: 31-Oct-2017 Date Received: 31-Jan-2018
Final Report for Period Beginning 07-Jul-2014 and Ending 31-Jul-2017
Title: Studies in the Control of Stochastic Systems
Begin Performance Period: 07-Jul-2014 **End Performance Period:** 31-Jul-2017
Report Term: 0-Other
Submitted By: Tyrone Duncan Email: duncan@ku.edu
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Distribution Statement: 1-Approved for public release; distribution is unlimited.

STEM Degrees: 10 **STEM Participants:** 14

Major Goals: The research supported by this ARO grant has focused on the control of continuous time stochastic systems with noise that is Brownian motions or fractional Brownian motions, the control of discrete time stochastic systems with arbitrary correlated noise and stochastic differential games. In modeling physical systems the perturbations or the unmodeled dynamics are typically represented by an addditive noise perturbation of the mathematical model. Such modeling has been quite effective in a variety of physical systems. Some important examples are space exploration and telecommunications. Historically the continuous noise has been modeled by a Brownian motion which was identified in the physics literature in the beginning of the twentieth century by Einstein and Smoluchowski. However based on empirical data from many physical phenomena it has been verified that other noise models are often required. One family of stochastic models that have been identified empirically is the family of fractional Brownian motions. The fractional Brownian motions are a family of centered Gaussian processes indexed by the H8rst parameter, $H \in (0,1)$. The usefulness of these processes has been verified from empirical data as appropriate for models of rainfall, turbulence, economic data, cognition, telecommunications, and epileptic seizures. Most physical systems require stochastic models. Furthermore most physical systems are controlled. The strategies can be represented by one agent or by multi-agents. In the former case the problem can be posed as stochastic control and in the latter case the problem can be posed as stochastic differential games. The game problems have two or more competing agents and a payoff functional. These problems arise in natural resources allocation, financial systems and warfare. The stochastic control and game problems are formulated with a cost functional or a payoff functional to optimize and can evolve in a finite time horizon or an infinite time horizon. In the infinite time horizon setting the functional can have a discount factor or be a long run average (ergodic) criteria. Often the complete system state is not available to the controller so the system is termed partially observed. In this case the state of the system has to be estimated. The cost functionals and payoff functionals are typically quadratic in the state and the strategies or exponential quadratic to allow for explicit optimal strategies. Many stochastic systems require modeling as stochastic partial differential equations so some control problems and games are formulated in this setting. Typical partial differential equations are obtained from the heat equation or the wave equation which can be used to model many distributed systems. The stochastic partial differential equations are described by stochastic equations in an infinite dimensional Hilbert space. These stochastic partial differential equations can be driven by Brownian motions or fractional Brownian motions. The control or game problems in this infinite dimensional setting have usually quadratic or exponential quadratic cost or payoff

functionals and these results have been investigated. These optimization problems can be solved explicitly for optimal strategies. Often stochastic systems evolve in nonlinear spaces that have some differential geometric description. Some examples of these spaces that are called symmetric spaces are unit spheres or open unit balls in arbitrary dimensional Euclidean spaces. These problems have been investigated. With suitable cost criteria these problems can be explicitly solved for both control and game problems and this has been done. Thus finite and infinite dimensional stochastic systems with a variety of noise processes are considered to solve control and differential game problems in both continuous and discrete time. All of the above types of problems have been studied with the support of this grant.
The achievement of these goals can provide some major contributions to the scientific base of the United States. Some applications of these results can be important for contributing to important applications that can contribute to the industrial development of the United States.

Accomplishments: A PDF document has been uploaded in the Upload Section.

Training Opportunities: During the grant period one student completed his doctoral study, nine students completed their masters degrees and one student completed his undergraduate honors thesis.

Results Dissemination: Workshops for high school students at international conferences.
a) IFAC World Congress Cape Town, SA Aug. 2014.
b) CDC2014 Los Angeles, Dec. 2014.
c) ACC2015 Chicago, July 2015, celebrated fifteen years of workshops at ACCs lead by BPD.
d) ACC2016 Boston, July 2016.
e) CDC2016 Las Vegas, Dec. 2016.
f) ACC2017 Seattle May 2017.
g) IFAC World Congress Toulouse, France 9-14 July 2017. Panel sessions by BPD were arranged by a request from the Congress President for a full day (Wednesday) during the Congress. These sessions focused on sharing research with a broader audience and preparing future scientists and engineers.
2. Math Awareness Months (MAM) (Every April for the past twenty-three years)
Agenda: workshops each year for fifth graders from two schools on different days, math competitions on the first Saturday of April, lectures for a broad audience, MAM declarations from city and state.
Math Competitions at three levels: 3-6 grades, 8-9 grades and 9-12 grades: the local schools, the schools in Kansas City and Topeka are well represented, more than 80 schools throughout Kansas, total number of participants has been at least 150 for a number of years; many students have come for consecutive years.

Honors and Awards: 1. SIAM Reid Prize (TD) 7/13 (unique awardee)
2. SIAM Fellow (TD) 3/15 (1 of 31)
3. Simons Fellow 2015 (TD) 8/15-5/16 (1 of 40 mathematicians from US, Canada and UK)
4. IFAC Fellow (BPD) 8/14 (1 of 32 for period 2011-2013)
5. Chancellors Club Teaching Professorship (BPD) 8/15 (unique awardee)
6. IEEE Educational Activities Board Meritorious Achievement Award in Continuing Education for "innovative developments in teaching control systems and inspiring STEM education" (BPD) 6/16 (unique awardee)
7. 2017 IFAC Outstanding Service Award (BPD) (unique award)
8. Elected Global Chair of IEEE Women in Engineering (BPD) 2017 and reelected for 2018 (more than 20,000 members)

Protocol Activity Status:

Technology Transfer: Nothing to Report

PARTICIPANTS:

Participant Type: PD/PI

Participant: Tyrone Edward Duncan
Person Months Worked: 9.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

Participant Type: Co PD/PI
Participant: Bozenna Janina Pasik Duncan
Person Months Worked: 4.00 **Funding Support:**
Project Contribution:
International Collaboration:
International Travel:
National Academy Member: N
Other Collaborators:

CONFERENCE PAPERS:

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: IEEE Conference on Decison and Control
Date Received: 14-Sep-2016 Conference Date: 15-Dec-2015 Date Published: 14-Dec-2015
Conference Location: Osaka, Japan
Paper Title: Some stochastic differential games with state dependent noise
Authors: Tyrone Duncan, Bozenna Pasik-Duncan
Acknowledged Federal Support: **Y**

Publication Type: Conference Paper or Presentation **Publication Status:** 1-Published
Conference Name: 11th IFAC Symp. On Advances in Control Education
Date Received: 14-Sep-2016 Conference Date: 13-Jun-2016 Date Published: 13-Jun-2016
Conference Location: Bratislava
Paper Title: Stochastic Adaptive Control - Integrating Research and Teaching
Authors: Tyrone Duncan, Bozenna Pasik-Duncan
Acknowledged Federal Support: **Y**

1 Accomplishments

The goals proposed for this study have been largely achieved and in some cases extended beyond the problems proposed. A direct method for solving stochastic control and stochastic differential games has been developed so that solutions of control or game problems are obtained without the requirements of solving nonlinear partial differential equations (Hamilton- Jacobi-Bellman or Hamilton-Jacobi-Isaacs equations) or using a stochastic maximum principle with backward stochastic differential equations. This direct method can be used to solve both continuous and discrete time control problems.

One type of problem that has been solved is a stochastic differential game with a general square integrable noise process and a quadratic payoff that is described now.

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) \qquad (1) \\ X(0) &= X_0 \end{aligned}$$

where $X_0 \in \mathbb{R}^n$ is not random, $X(t) \in \mathbb{R}^n, A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $U(t) \in \mathbb{R}^m$, $U \in \mathcal{U}$, $C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$, $V(t) \in \mathbb{R}^p$, $V \in \mathcal{V}$, and $F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$. The terms U and V denote the control actions of the two players. The positive integers (m, n, p, q) are arbitrary. The process $(W(t), t \geq 0)$ is a square integrable stochastic process with continuous sample paths that is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for W. The family of admissible strategies for U is $\mathcal{U}$ and for V is $\mathcal{V}$ and they are defined as follows

$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $U \in L^2([0, T])$ a.s.$\}$

and

$\mathcal{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $V \in L^2([0, T])$ a.s.$\}$

The cost functional J is a quadratic functional of X, U, and V that is

given by

$$
\begin{aligned}
J^0(U,V) &= \frac{1}{2}[\int_0^T (< QX(s), X(s) > + < RU(s), U(s) > \\
&\quad - < SV(s), V(s) >)ds + < MX(T), X(T) >] \\
J(U,V) &= \mathbb{E}[J^0(U,V)] \qquad (2)
\end{aligned}
$$

where $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $Q > 0$, $R > 0$, $S > 0$, and $M \geq 0$ are symmetric linear transformations. The following theorem provides an explicit solution to this noncooperative two person linear quadratic game with a general noise process, W. It seems that there are no other results available for these games when W is an arbitrary stochastic process with continuous sample paths.

Theorem 1.1. *The two person zero sum stochastic differential game given by* (**??**) *and* (**??**) *has optimal admissible strategies for the two players, denoted* U^* *and* V^**, given by*

$$
\begin{aligned}
U^*(t) &= -R^{-1}(B^T P(t)X(t) + B^T \hat{\phi}(t)) \qquad (3) \\
V^*(t) &= S^{-1}(C^T P(t)X(t) + C^T \hat{\phi}(t)) \qquad (4)
\end{aligned}
$$

where $(P(t), t \in [0,T])$ *is the unique positive solution of the following equation*

$$
\begin{aligned}
-\frac{dP}{dt} &= Q + PA + A^T P - P(BR^{-1}B^T - CS^{-1}C^T)P \qquad (5) \\
P(T) &= M \qquad (6)
\end{aligned}
$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T > 0$ *and* $(\phi(t), t \in [0,T])$ *is the solution of the following linear stochastic equation*

$$
\begin{aligned}
d\phi(t) &= -[(A^T - P(t)BR^{-1}B^T + P(t)CS^{-1}C^T)\phi dt \qquad (7) \\
&\quad + P(t)FdW(t)] \\
\phi(T) &= 0 \qquad (8)
\end{aligned}
$$

and

$$
\hat{\phi}(t) = \mathbb{E}[\phi(t)|\mathcal{F}(t)] \qquad (9)
$$

This problem has been generalized in various ways. The linear system has been generalized to an infinite dimensional Hilbert space to model stochastic

partial differential equations where the operator A is the generator of a C_0 semigroup and the noise is a cylindrical fractional Brownian motion.

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + CV(t)dt + \Phi dW(t) \\ X(0) &= x \end{aligned} \tag{10}$$

where $X(t) \in \mathcal{H}$ for $t \in [0,T]$, $X_0 \in \mathcal{H}$, $\mathcal{H}$ is a real, separable, infinite dimensional Hilbert space, $A : D_A \subset \mathcal{H} \to \mathcal{H}$ is a linear and (in general) unbounded operator that is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$, $U(t) \in \mathcal{U}, V(t) \in \mathcal{V}$ for Hilbert spaces $\mathcal{U}$ and $\mathcal{V}$. $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the family of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. The process $(W(t), t \in [0,T])$ is a standard cylindrical fractional Brownian motion in $\mathcal{H}$ with the Hurst parameter $H \in (\frac{1}{2}, 1)$ fixed and it is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0,T])$ is the filtration for W on this probability space. The linear operator A is usually the infinitesimal generator of an analytic semigroup on $\mathcal{H}$ so that for some $\hat{\beta} > 0$ the operator $-A + \hat{\beta} I$ is strictly positive so that the fractional powers $(-A + \hat{\beta} I)^{\gamma}$ and $(-A^* + \hat{\beta} I)^{\gamma}$ and the spaces $D_A^{\gamma} = D((-A + \hat{\beta} I)^{\gamma})$ and $D_{A^*}^{\gamma} = D((-A^* + \hat{\beta} I)^{\gamma})$ with the graph norm topology for $\gamma \in \mathbb{R}$ can be defined. The linear space $D(\cdot)$ denotes the domain of $\cdot$. The linear operators B, C and Φ and the family of admissible strategies, $(\mathcal{U}_a, \mathcal{V}_a)$, for the two players satisfy natural conditions.
An ergodic payoff is considered.
The payoff for a $T > 0$ is

$$\begin{aligned} J_T^0(U,V) &= \frac{1}{2}[\int_0^T (< QX(t), X(t) \\ &\quad + < RU(t), U(t) > - < SV(t), V(t) >)dt] \end{aligned} \tag{11}$$

$$J_T(U,V) = \mathbb{E} J_T^0(U,V) \tag{12}$$

The family of admissible strategies for U is $\mathcal{U}$ and for V is $\mathcal{V}$ and they are defined as follows

$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is $(\mathcal{F}(t), t \in [0,\infty))$ progressively measurable such that $U \in L^2([0,T])$ a.s. for each $T > 0\}$
and
$\mathcal{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is $(\mathcal{F}(t), t \in [0,\infty))$ progressively

measurable process such that $V \in L^2([0,T])$ a.s. for each $T > 0\}$

where $(\mathcal{F}(t), t \in [0,\infty))$ is the natural filtration for $(W(t), t \geq 0)$. The expected long run average (ergodic) cost for the stochastic game is

$$J^g_\infty(U,V) = lim\ sup_{T\to\infty} \frac{1}{T} J_T(U,V)$$

The formal Riccati equation for this stochastic game which can be precisely defined is

$$\begin{aligned} \frac{dP}{dt} &= A^*P + PA - P(BR^{-1}B^* - CS^{-1}C^*)P + Q \qquad (13) \\ P(0) &= G \end{aligned}$$

Theorem 1.2. *Let some natural assumptions be satisfied. For the stochastic differential game given by* (**??**) *and* (**??**) *and the admissible strategies* $\mathcal{U}_a$ *and* $\mathcal{V}_a$ *there are optimal strategies,* U^* *and* V^**, given by*

$$U^*(t) = -R^{-1}B^*(P(t)X(t) + \hat{\varphi}(t)) \qquad (14)$$

$$V^*(t) = S^{-1}C^*(P(t)X(t) + \hat{\varphi}(t)) \qquad (15)$$

where

$$\begin{aligned} d\varphi(t) &= -[(A^* - P(t)BR^{-1}B^* + P(t)CS^{-1}C^*)\varphi dt \qquad (16) \\ &\quad +P(t)\Phi dW(t)] \\ \varphi(T) &= 0 \qquad (17) \end{aligned}$$

and

$$\hat{\varphi}(t) = \mathbb{E}[\varphi(t)|\mathcal{F}(t)] \qquad (18)$$

The optimal payoff for this Nash equilibrium can be computed directly.

Some nonlinear stochastic differential games can be explicitly solved. The following game evolves in the unit sphere in Euclidean three-space. The stochastic differential game is described by the following equation which describes the distance of the process, $(Y(t), t \in [0,T])$, from a point on the sphere denoted o which is called the origin for the differential game.

$$\begin{aligned} dX(t) &= \frac{1}{2} cot \frac{X(t)}{2} dt + bU(t)dt + cV(t)dt \qquad (19) \\ &+ dB(t) \\ X(0) &= X_0 \qquad (20) \end{aligned}$$

where $Y(t) \in S^2 \setminus A_o$, where A_0 is the antipodal manifold, $X(t) = |Y(t)|$, $(B(t), t \in [0,T])$ is a real-valued standard Brownian motion for a fixed $T > 0$, $X_0 \in (0, L)$ is a constant and L is the distance from o to A_o. The Brownian motion is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0,T])$ is the filtration for the Brownian motion B. The terms (b, c) are nonzero real numbers. An assumption on the relative size of these two real numbers is made subsequently. The family of admissible control strategies for U is $\mathcal{U}$ and for V is $\mathcal{V}$ and they are defined as follows
$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0,T])$ such that $U \in L^2([0,T])$ a.s.$\}$
and
$\mathcal{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0,T])$ such that $V \in L^2([0,T])$ a.s.$\}$
If $U(t)$ and $V(t)$ are suitably smooth functions of $X(t)$, then $(X(t), t \in [0,T])$ is a Markov process with the infinitesimal generator

$$\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{2}cot(\frac{r}{2})\frac{\partial}{\partial r} + U(r)\frac{\partial}{\partial r} + V(r)\frac{\partial}{\partial r} \tag{21}$$

The payoff for the stochastic differential game with the control strategies U and V is denoted $J_T(U,V)$ that is described as follows

$$\begin{aligned} J_T^0(U,V) &= \int_0^T (a\ sin^2\frac{X(t)}{4} + U^2(t)cos^2\frac{X(t)}{4} \\ &- V^2(t)cos^2\frac{X(t)}{4})dt \qquad (22) \\ J_T(U,V) &= \mathbb{E}J_T^0(U,V) \qquad (23) \end{aligned}$$

The following scalar Riccati and linear equations are used in the solution of the control problem.

$$\begin{aligned} \frac{dg(t)}{dt} &= \frac{3}{8}g + \frac{1}{16}g^2(b^2 - c^2) - a \qquad (24) \\ g(T) &= 0 \qquad (25) \\ \frac{dh(t)}{dt} &= -\frac{3}{16}g \qquad (26) \\ h(T) &= 0 \qquad (27) \end{aligned}$$

Theorem 1.3. *Let $b^2 > c^2 > 0$. The stochastic differential game described by (??) and (??) has optimal control strategies, (U^*, V^*), for the two players*

that are given by

$$\begin{aligned} U^*(t) &= -\frac{b}{4} g(t) tan \frac{X(t)}{4} \\ V^*(t) &= \frac{c}{4} g(t) tan \frac{X(t)}{4} \end{aligned}$$

where $t \in [0, T]$ and g is the positive solution of the scalar Riccati equation. The value of the game is

$$J_T(U^*, V^*) = g(0) sin^2 \frac{X(0)}{4} + h(0)$$

where h is given by (**??**).

A similar approach can be used to verify optimal control strategies for some infinite time horizon problems with a long run average payoff for the stochastic differential game. In this case the payoff is

$$J_\infty(U, V) = lim\ sup_{T \to \infty} \frac{1}{T} J_T(U, V)$$

where J_T is given above. The payoff function for these game problems has the important property that $sin^2 \frac{x}{4}$ is an eigenfunction of the radial part of the Laplacian for S^2. Other payoff functionals can be defined by choosing other eigenfunctions of the radial part of the Laplacian. Similarly stochastic differential games in the n-sphere and other compact rank one symmetric spaces can be formulated and explicitly solved.

The following control problem is for a linear equation in a Hilbert space with an additive fractional Brownian motion and an ergodic quadratic cost. An infinite horizon control problem is described by a long term average or ergodic quadratic cost functional.

$$\begin{aligned} dX(t) &= (AX(t) + Bu(t))dt + dB_H(t) && (28) \\ X(0) &= x && (29) \end{aligned}$$

where $x \in V$, $X(t) \in V$, V is an infinite dimensional real separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $| \cdot |$. The process $(B_H(t), t \geq 0)$ is a V-valued fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$

and having the incremental covariance $\widetilde{Q}$ where $\widetilde{Q}$ is trace class ($Tr(\widetilde{Q}) < \infty$) so that

$$\mathbb{E} < B_H(t), x >< B_H(s), y >= \frac{1}{2} < \widetilde{Q}x, y > (t^{2H} + s^{2H} - |t-s|^{2H}). \quad (30)$$

for $x, y \in V$. The operator $A : Dom(A) \to V$ with $Dom(A) \subset V$ is a linear, densely defined operator on V which is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$. Let $\mathcal{U} = (U, < \cdot, \cdot >_U, |\cdot|_U)$ be another Hilbert space, the state space of controls, and assume that $B \in \mathcal{L}(U, V)$. Furthermore consider the family of admissible controls, $\mathcal{U}$, defined as follows

$$\mathcal{U} = \{u : \mathbb{R}_+ \times \Omega \to U, u \text{ is progressively measurable,} \\ \mathbb{E}\int_0^T |u(t)|_U^2 dt < \infty \text{ for all } T > 0\}$$

The solution of the equation (**??**) is defined as the mild solution, that is,

$$X(t) = S(t)x + \int_0^t S(t-s)Bu(t)dt + \int_0^t S(t-s)dB_H(t) \quad (31)$$

for $t \geq 0$ and it is known that with the above assumptions there is one and only one V- continuous solution to (**??**) . Now the cost functional is defined for the control problem. Let J_T be given as follows

$$J_T(x, u) := \frac{1}{2}\int_0^T (|LX(s)|^2 + < Ru(s), u(s) >_U)ds \quad (32)$$

where $L \in \mathcal{L}(V)$, $R \in \mathcal{L}(U)$, R is self-adjoint and invertible. The control problem is to minimize the following ergodic cost

$$lim\ sup_{T\to\infty}\frac{1}{T}\mathbb{E}J_T(x, u). \quad (33)$$

Theorem 1.4. *Let detectability and stabilizability conditions be satisfied and let $u \in \mathcal{U}$ be a control satisfying*

$$lim_{T\to\infty}\frac{1}{T}\mathbb{E} < PX^u(T), X^u(T) >= 0 \quad (34)$$

*where $(X^u(T), T \in [0, \infty))$ is the solution to (**??**) with the control $u \in \mathcal{U}$. Then*

$$lim\ sup_{T\to\infty}\frac{1}{T}\mathbb{E}J_T(x, u) \geq J_\infty \quad (35)$$

where

$$J_\infty := lim\ sup_{T\to\infty}\frac{1}{2T}\mathbb{E}\int_0^T |R^{\frac{1}{2}}B^*V(s)|_U^2 ds \tag{36}$$

$$+\int_0^\infty Tr(\widetilde{Q}P\Phi(t))\phi_H(r)dr \tag{37}$$

for each $x \in V$ where $\phi(r) = H(2H-1)|r|^{2H-2}, r \in \mathbb{R}$. Moreover, the feedback control $\hat{u}(t) = -R^{-1}B^*(PX^{\hat{u}}(s) + V(s))$ is admissible, satisfies the condition (**??**) *and*

$$lim\ sup_{T\to\infty}\frac{1}{T}\mathbb{E}J_T(x,\hat{u}) = J_\infty \tag{38}$$

for each $x \in V$. Thus $\hat{u}$ is an optimal ergodic control and J_∞ is the optimal cost for the ergodic control problem ((**??**)*-*(**??**)*).*

Consider the following controlled linear stochastic system with a general noise process

$$dX(t) = AX(t)dt + CU(t)dt + dB(t) \tag{39}$$

$$X(0) = X_0 \tag{40}$$

where $X_0 \in \mathbb{R}^n$ is not random, $X(t) \in \mathbb{R}^n, A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), C \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $U(t) \in \mathbb{R}^m$, $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^l)$ denotes the family of linear transformations from $\mathbb{R}^k$ to $\mathbb{R}^l$, $U \in \mathcal{U}$, $(B(t), t \in [0,T])$ is an $\mathbb{R}^n$-valued zero mean, square integrable process with continuous sample paths with $B(0) = 0$ and this process is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $T > 0$ is fixed.

The family of nonadapted admissible controls, $\mathcal{U}$, is
$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process such that $U \in L^2([0,T])$ a.s.$\}$
Let $(\mathcal{F}(t), t \in [0,T])$ be the filtration of $(B(t), t \in [0,T])$. The family of adapted, admissible controls, $\mathcal{U}_a$, is
$\mathcal{U}_a = \{U : U$ is an $\mathbb{R}^m$-valued $(\mathcal{F}(t), t \in [0,T])$ progressively measurable process such that $U \in L^2([0,T])$ a.s.$\}$
The cost functional J is a quadratic functional of X and U that is given by

$$\begin{aligned} J(U) &= \frac{1}{2}E[\int_0^T < QX(s), X(s) > + < RU(s), U(s) > ds] \\ &+\frac{1}{2}E < MX(T), X(T) > \end{aligned} \tag{41}$$

where $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $Q > 0$, $R > 0$ and $M \geq 0$ are symmetric linear transformations and $< \cdot, \cdot >$ denotes the canonical Euclidean inner product on the Euclidean space of the appropriate dimension. The dependence of J on the initial condition X_0 is suppressed for notational convenience.

Theorem 1.5. *For the optimal control problem* (**??**) *and* (**??**) *and the family of admissible, nonadapted controls, $\mathcal{U}$, there is an optimal control U^* that can be expressed as*

$$U^*(t) = -R^{-1}C^T(P(t)X(t) + W(t)) \tag{42}$$

where $(P(t), t \in [0, T])$ is the unique symmetric positive definite solution of the Riccati equation

$$\begin{aligned} \frac{dP}{dt} &= -PA - A^T P + PCR^{-1}C^T P - Q & (43)\\ P(T) &= M & (44) \end{aligned}$$

and $(W(t), t \in [0, T])$ is the process that satisfies

$$W(t) = \int_t^T \Phi_P(s, t)P(s)dB(s) \tag{45}$$

and B is the process in (**??**) *and Φ_P is the fundamental solution of the matrix equation*

$$\begin{aligned} \frac{d\Phi_P(s,t)}{dt} &= -(A^T - P(t)CR^{-1}C^T)\Phi_P(s,t) & (46)\\ \Phi_P(s,s) &= I & (47) \end{aligned}$$

Corollary 1.6. *Let $(B(t), t \in [0, T])$ in* (**??**) *be a standard fractional Brownian motion with a fixed Hurst parameter $H \in (0, 1)$. For the optimal control problem given by* (**??**) *and* (**??**) *and the family of admissible nonadapted controls $\mathcal{U}$ an optimal control U^* is given by*

$$\bar{U}^*(t) = R^{-1}C^T(P(t)X(t) + W(t)) \tag{48}$$

where $(P(t), t \in [0, T])$ is the unique positive definite symmetric solution of (**??**) *and $(W(t), t \in [0, T])$ is the process that satisfies*

$$W(t) = \int_t^T \Phi_P(s, t)P(s)dB(s) \tag{49}$$

*and Φ_P is the solution of (**??**). For $H \in (\frac{1}{2}, 1)$ the optimal cost is*

$$\begin{aligned} J(U^*) &= \frac{1}{2} < P(0)X_0, X_0 > \\ &-\frac{1}{2}\int_0^T \int_t^T \int_t^T tr(P(r)\Phi_P^T(r,t)CR^{-1}C^T\Phi_P(s,t)P(s)) \\ &\times \phi_H(s-r)drdsdt + \int_0^T \int_s^T tr(\Phi_P(s,t)P(s)) \\ &\times \phi_H(s-t)dsdt \end{aligned} \tag{50}$$

where $\phi_H(s) = H(2H-1)|s|^{2H-2}$.

Now a partially observed control problem is described with a risk sensitive cost functional. Initially the system and the observation equations are described. The equation for the system process X is given by

$$\begin{aligned} dX(t) &= (AX(t) + CU(t))dt + FdB(t) \\ X(0) &= X_0 \end{aligned} \tag{51}$$

where X_0 is a constant vector in $\mathbb{R}^n$, $X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^m, A \in L(\mathbb{R}^n, \mathbb{R}^n), C \in L(\mathbb{R}^m, \mathbb{R}^n), F \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $(B(t), t \geq 0)$ is an $\mathbb{R}^n$-valued standard Brownian motion. The process B is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The observation process $(Y(t), t \in [0,T])$ satisfies the following stochastic equation

$$\begin{aligned} dY(t) &= HX(t)dt + GdV(t) \\ Y(0) &= 0 \end{aligned} \tag{52}$$

where $Y(t) \in L(\mathbb{R}^p)$, $H \in L(\mathbb{R}^n, \mathbb{R}^p)$, $G \in L(\mathbb{R}^p, \mathbb{R}^p)$ is invertible and $(V(t), t \geq 0)$ is an $\mathbb{R}^p$-valued standard Brownian motion that is also defined on $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that the processes B and V are independent. Let $(\mathcal{G}(t), t \in [0,T])$ be the natural filtration for the process $(Y(t), t \in [0,T])$ on $(\Omega, \mathcal{F}, \mathbb{P})$. The family of admissible controls, $\mathcal{U}$, is defined as

$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued $(\mathcal{G}(t), t \in [0,T])$ progressively measurable process such that $U \in L^2([0,T])$ a.s.$\}$

The cost, $J(\cdot)$, is an exponential quadratic functional of the state and the

control that is given as follows

$$
\begin{aligned}
J(U) &= \mu \mathbb{E} \exp[\frac{\mu}{2} \int_0^T (< QX(s), X(s) > \\
&\quad + < RU(s), U(s) >) ds + \frac{\mu}{2} < MX(T), X(T) >]
\end{aligned} \tag{53}
$$

where $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ and $M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ are symmetric linear transformations, such that $Q > 0$, $R > 0$, $M \geq 0$ and μ is fixed. For the verification of an optimal control in this paper it is assumed that $M = 0$. Some remarks are made later about this restriction and how to eliminate it. The appropriate estimation equation, often called the information filter, is given by

$$
\begin{aligned}
dZ(t) &= (A - P(t)H^T H + \mu P(t) Q) Z(t) dt \\
&\quad + CU(t) dt + P(t) H^T dY(t) \\
Z(0) &= X(0)
\end{aligned} \tag{54}
$$

and $(P(t), t \in [0, T])$ is the unique, positive symmetric solution of the following Riccati equation

$$
\begin{aligned}
\frac{dP}{dt} &= AP + PA^T \\
&\quad - P(H^T H - \mu Q + FF^T) P \\
P(0) &= 0
\end{aligned} \tag{55}
$$

It is assumed that μ is chosen to satisfy $(H^T H - \mu Q + FF^T) > 0$. The process $(\int_0^t PH^T(dY - HZds), \mathcal{G}(t), t \in [0, T])$ is a Brownian motion by the Riccati equation (**??**) and sn absolute continuity result It follows from the results for the information filter that for observation measurable actions on the exponential quadratic cost, that it suffices to consider the process $(Z(t), t \in [0, T])$ because this process is the minimizing solution of the best estimate for the exponential of the quadratic form in X formed using Q. Thus the control for (**??**) is a function of the process Z. This estimate Z is given as follows

$$
\begin{aligned}
Z(\cdot) &= arg\ min_{h \in \mathcal{H}} \mathbb{E}[\mu exp(\frac{\mu}{2} \int_0^t < Q(X(s) - h(s)), \\
&\quad X(s) - h(s) > ds | \mathcal{G}(t)]
\end{aligned} \tag{56}
$$

where $\mathcal{H}$ is the family of square integrable $\mathcal{G}(\cdot)$ progressively measurable processes on $[0, T]$

Theorem 1.7. *For the control problem given by the state equation (??), the observation equation (??), and the cost functional (??) there is an optimal control, U^*, from the family of admissible controls, $\mathcal{U}$, that is given by*

$$U^*(t) = -R^{-1}C^T S(t)Z(t) \tag{57}$$

where $(S(t), t \in [0,T])$ is the unique positive, symmetric solution of the following Riccati equation

$$\begin{aligned} -\frac{dS}{dt} &= S(A + \mu PQ) + (A^T + \mu QP)S + Q \\ &\quad -(S(CR^{-1}C^T - \mu PH^T(GG^T)^{-1}HP)S \\ S(T) &= 0 \end{aligned} \tag{58}$$

Consider a financial market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $T < \infty$, satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_T$. Without loss of generality it is assumed that the savings account is constant and identically equal to one. Moreover, it is assumed that the price X of the underlying asset has a stochastic volatility given by a function of a standard fractional Brownian motion, so the dynamics of X is given by

$$dX(t) = f(W^H(t))g(t)X(t)\ dW(t), \tag{59}$$

where $X(0)$ is a positive constant, the process W is a standard Brownian motion, W^H is a standard fractional Brownian motion with the Hurst parameter $H \in (0,1)$, $f : \mathbb{R} \to \mathbb{R}^+$ is Borel measurable and $g : [0,T] \to \mathbb{R}^+$ is Borel measurable and bounded.

Theorem 1.8. *Let $t \in [0,T]$, X be given by (??), W^H the fractional Brownian motion and W, $\widehat{W}$ be correlated Brownian motions, $d\left\langle W, \widehat{W}\right\rangle_t = \rho(t)dt$ with a measurable, deterministic function $\rho : [0,T] \to (-1,1)$. The random variable X_t has a probability density function h_{X_t} satisfying*

$$h_{X(t)}(s) = \mathbb{E}\left[\frac{1}{s\sigma_H}\varphi\left(\frac{\ln\frac{s}{X_0} - \int_0^t f(W^H(u))g(u)\rho(u)d\widehat{W}(u) + 1/2\int_0^t f^2(W^H(u))g^2(u)du}{\sigma_H}\right)\right], \tag{60}$$

where $s > 0$, φ is the probability density of a standard Gaussian random variable $N(0,1)$, and

$$\sigma_H^2 = \int_0^t f^2(W^H(u))g^2(u)(1-\rho^2(u))du. \tag{61}$$

A two-person stochastic differential game with a risk sensitive quadratic payoff is considered now. The two person stochastic differential game is described by the following linear stochastic differential equation

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) && (62) \\ X(0) &= X_0 && (63) \end{aligned}$$

where $X_0 \in \mathbb{R}^n$ is not random, $X(t) \in \mathbb{R}^n, A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $U(t) \in \mathbb{R}^m$, $U \in \mathcal{U}$, $C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$, $V(t) \in \mathbb{R}^p$, $V \in \mathcal{V}$, and $F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$. The positive integers (m, n, p, q) are arbitrary. The process $(W(t), t \geq 0)$ is an $\mathbb{R}^q$ valued standard Brownian motion that is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for W. The terms U and V denote the strategies of the two players and the family of admissible strategies for U is $\mathcal{U}$ and for V is $\mathcal{V}$ and these families are defined as follows

$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $U \in L^2([0, T])$ a.s.$\}$
and
$\mathcal{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $V \in L^2([0, T])$ a.s.$\}$
The payoff J_μ is the exponential of a quadratic functional of X, U, and V that is given by

$$\begin{aligned} J^0_\mu(U, V) &= \mu exp[\frac{\mu}{2} \int_0^T (< QX(s), X(s) > && (64) \\ &+ < RU(s), U(s) > - < SV(s), V(s) >)ds \\ &+ \frac{\mu}{2} < MX(T), X(T) >] \\ J_\mu(U, V) &= \mathbb{E}[J^0_\mu(U, V)] && (65) \end{aligned}$$

where $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $Q > 0$, $R > 0$, $S > 0$, and $M \geq 0$ are symmetric linear transformations and $\mu \neq 0$ is fixed. An assumption on the possible values for μ is given in the following theorem. The player with control U seeks to minimize the payoff J_μ while the player with control V seeks to maximize the payoff J_μ.

Theorem 1.9. *The two person zero sum stochastic differential game described by* (**??**) *and* (**??**) *has a Nash equilibrium using the optimal admissible*

control strategies for the two players, denoted U^* and V^*, given by

$$U^*(t) = -R^{-1}B^T P(t)X(t) \tag{66}$$
$$V^*(t) = S^{-1}C^T P(t)X(t) \tag{67}$$

where $(P(t), t \in [0,T])$ is the unique positive symmetric solution of the following Riccati equation

$$\begin{aligned} -\frac{dP}{dt} &= Q + PA + A^T P \qquad (68) \\ &- P(BR^{-1}B^T - CS^{-1}C^T - \mu FF^T)P \\ P(T) &= M \qquad (69) \end{aligned}$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T - \mu FF^T > 0$. The optimal payoff is

$$J_\mu(U^*, V^*) = \mu exp[\frac{\mu}{2}(< P(0)X_0, X_0 > + \int_0^T tr(PFF^T)dt)] \tag{70}$$

A stochastic differential game problem that is formulated and solved to control the roots of a process in the Lie algebra $su(3)$ is now described. Since $SU(3)$ is simply connected, this game problem can be viewed in the Lie algebra, $su(3)$. The group $SU(3)$ has particular interest in physics because the Gell-Mann matrices are generators for $SU(3)$ that mediate Quantum Chromodynamics (QCD) which is also known as the Strong Force. In theoretical physics QCD is the theory of strong interactions that is a fundamental force describing the interactions between quarks and gluons which comprise hadrons such as the proton, neutron and pion. This theory is an important part of the Standard Model of particle physics.
The simply connected Lie group $SU(3)$ is the family of 3×3 unitary matrices with determinant one, that is, $g \in SU(3)$ if $gg^* = I$, $det(g) = 1$. This Lie group has dimension eight as a real manifold. It is a simple Lie group. This Lie group has rank two, that is, the dimension of the Cartan subalgebra is two.

The stochastic differential game is described by a stochastic differential equation that has terms from the strategies of the two players and terms

from the radial part of the Laplacian.

$$
\begin{aligned}
dX_1(t) &= \frac{1}{2}(c\coth X_1(t)) + b\coth\frac{X_1(t)}{2} + a\frac{\sinh X_1(t)}{\cosh X_2(t) - \cosh X_1(t)})dt \\
&+ \alpha U_1(t)dt + \beta V_1(t)dt + dB_1(t) \qquad (71) \\
dX_2(t) &= \frac{1}{2}(c\coth X_2(t) + b\coth\frac{X_2(t)}{2} + a\frac{\sinh X_2(t)}{\cosh X_1(t) - \cosh X_2(t)})dt \\
&+ \alpha U_2(t)dt + \beta V_2(t)dt + dB_2(t) \qquad (72) \\
X_1(0) &= x_{10} \qquad (73) \\
X_2(0) &= x_{20} \qquad (74)
\end{aligned}
$$

The process $((B_1(t), B_2(t)), t \in [0, T])$ is an $\mathbb{R}^2$-valued standard Brownian motion that is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for the Brownian motion (B_1, B_2), x_{10} and x_{20} are constants and α, β are strictly positive constants. Player I has the control pair (U_1, U_2) and player II has the control pair (V_1, V_2). It is assumed that the positive real numbers α, β satisfy $\alpha^2 - \beta^2 > 0$. The symmetry of the two scalar equations for X_1 and X_2 is inherited from the coordinate symmetry for the radial part of the Laplacian. The payoff functional, $J(U, V)$, is

$$
\begin{aligned}
J^0(U,V) &= \int_0^T (\sinh^2\frac{X_1(t)}{2} + \sinh^2\frac{X_2(t)}{2} + (U_1^2(t) - V_1^2(t))\cosh^2\frac{X_1(t)}{2} \\
&\quad + (U_2^2(t) - V_2^2(t))\cosh^2\frac{X_2(t)}{2})dt \qquad (75) \\
J(U,V) &= \mathbb{E}J^0(U,V) \qquad (76)
\end{aligned}
$$

Theorem 1.10. *The stochastic differential game given by (**??**), (**??**), and (**??**) has the following optimal strategies, (U^*, V^*), that form a Nash equilibrium*

$$
\begin{aligned}
U_1^*(t) &= -\frac{1}{2}\alpha g(t)\tanh\frac{X_1(t)}{2} \qquad (77) \\
U_2^*(t) &= -\frac{1}{2}\alpha g(t)\tanh\frac{X_2(t)}{2} \qquad (78) \\
V_1^*(t) &= \frac{1}{2}\beta g(t)\tanh\frac{X_1(t)}{2} \qquad (79) \\
V_2^*(t) &= \frac{1}{2}\beta g(t)\tanh\frac{X_2(t)}{2} \qquad (80)
\end{aligned}
$$

The optimal payoff is

$$J(U^*, V^*) = g(0)(\sinh^2 \frac{x_{10}}{2} + \sinh^2 \frac{x_{20}}{2}) + h(0) \tag{81}$$

A control problem is solved for a stochastic evolution equation with a state dependent noise process. The noise can be a fractional Brownian motion for the Hurst parameter in the interval $(\frac{1}{2}, 1)$ or some other noise processes. The controls are restricted to linear state feedback. Consider the stochastic evolution equation

$$\begin{aligned} dX(t) &= (A(t)X(t) + B(t)K(t)X(t))dt + \sigma(t)X(t)db(t) \\ X(0) &= x_0 \end{aligned}$$

in a separable real Hilbert space $V = (V, |\cdot|, \langle \cdot, \cdot \rangle)$ where $(b(t), t \geq 0)$ is a real-valued Gauss-Volterra noise that is described below, $(A(t), t \geq 0)$ is a family of closed, (in general) unbounded, operators on V such that $\mathrm{Dom}(A(t)) = \mathrm{Dom}(A(0))$ for each $t \in \mathbb{R}_+$, and $\mathrm{Dom}(A^*(t)) = \mathrm{Dom}(A^*(0))$, that generates a strongly continuous evolution operator $(U_0(t,s)R,\ 0 \leq s \leq t < \infty)$. Furthermore, denoting by $C_s([a,b]), \mathcal{L}(Y_1, Y_2))$ the family of strongly continuous mappings $[a,b] \to \mathcal{L}(Y_1, Y_2)$ where Y_1, Y_2 are Hilbert spaces, $B \in C_s(\mathbb{R}_+, \mathcal{L}(U, V))$ and $K \in C_s(\mathbb{R}_+, \mathcal{L}(V, U))$, where $(U = U, \langle \cdot, \cdot \rangle_V, |\cdot|_V)$ is another Hilbert space; the process $u(t) = K(t)X(t)$ is described as a linear feedback control of the system and some linear-quadratic control problems are studied in the subsequent sections. Finally, σ is a continuous real-valued function.

Some details concerning the real-valued driving process $(b(t), t \geq 0)$ are given now. The process $(b(t), t \geq 0)$ is a Gauss-Volterra process, which is described by the covariance

$$R(t,s) = \mathbb{E}b(t)b(s) := \int_0^{\min(t,s)} K(t,r)K(s,r)dr, \tag{82}$$

where the kernel $K : \mathbb{R}_+^2 \to \mathbb{R}$ satisfies some conditions.

The admissible controls are of the state feedback form $u(t) = K(t)X(t)$ where $K \in C_s([0,T], \mathcal{L}(V, U))$. The cost functional to be minimized is

$$J_T(K) := \mathbb{E}\int_0^T (|L(t)X(t)|^2 + \langle R(t)K(t)X(t), K(t)X(t)\rangle_U)dt + \mathbb{E}\langle GX(T), X(T)\rangle \tag{83}$$

where $L \in C_s([0,T], \mathcal{L}(V))$ and $G = G^*$, $G \in \mathcal{L}(V)$, $G \geq 0$, $R \in C_s([0,T], \mathcal{L}(U))$ is such that $R(t) = R^*(t)$, and for some $\gamma_0 > 0$, $\langle R(t)u, u \rangle_U \geq \gamma_0 |u|_U^2$, $u \in U$, $t \in [0,T]$.

The operator Riccati differential equation associated with this control problem is

$$\begin{aligned} &\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^*(t)P(t) + L^*(t)L(t) \\ &\qquad - \alpha(t)P(t) = 0, \ t \in (0,T], \\ &P(T) = G. \end{aligned} \tag{84}$$

Note that this Riccati equation is different from the Riccati equation for a linear-quadratic control problem.

The following result solves the finite time horizon problem.

Theorem 1.11. *Let conditions on the noise and the Riccati equation be satisfied. Then the feedback control $u(t) = -R^{-1}B^*(t)P(t)X(t)$ is optimal for the control problem that is, the operator function $\hat{K}(t) = R^{-1}(t)B^*(t)P(t)$ minimizes J_T on the space of all $K \in C_s([0,T], \mathcal{L}(V,U))$. The optimal cost is*

$$J_T(\hat{K}) = \langle P(0)x_0, x_0 \rangle. \tag{85}$$